Supplementary Notes for ELEN 4810 Lecture 1
DT Signals + A Brief Review of Complex Arithmetic

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Disclaimer: These notes are intended to be an accessible introduction to the subject, with no pretense at completeness. In general, you can find more thorough discussions in the textbook. Please let me know if you find any typos.

Reading suggestions: Oppenheim and Schafer Chapter 1 and Sections 2.1-2.3

Optional (nongraded) homework suggestion: Spend some time pondering Euler's formula. Make a picture in Matlab to convince yourself that Euler's formula is true. Make sure you have a crisp understanding of the meaning of infinite summations. This is a technical point, but a clear understanding will be very helpful when we get to the $\mathcal{Z}$-transform.

1 Discrete-Time Signals

Throughout the course, we use

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\} \tag{1.1}$$

to denote the integers. Note that $\mathbb{Z}$ includes negative numbers. We use $\mathbb{R}$ for the real numbers, and $\mathbb{C}$ for the complex numbers. We will review the complex numbers in detail below.

A *discrete-time signal* is a sequence of numbers $x[n]$, one for each integer $n \in \mathbb{Z}$:

$$\dots,\ x[-2],\ x[-1],\ x[0],\ x[1],\ x[2],\ \dots. \tag{1.2}$$

The textbook occasionally uses the notation

$$\{x[n]\}_{n\in\mathbb{Z}} \tag{1.3}$$

to refer to the sequence as a whole; this notation is a bit cumbersome, but for consistency we will adopt it. Typically, we simply refer to "the sequence $x[n]$," in the same way that we might refer to "the function $f(t)$."

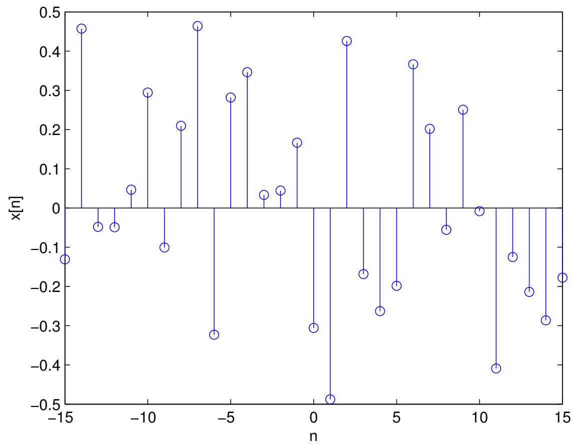

Figure 1: **A discrete-time signal** $x[n]$.

We can visualize a discrete-time signal via a stem plot, as in Figure 1. At each integer n, we have a value $x[n]$. We usually refer to the argument n as "time," although it could have other physical meanings.[1]

We will always use square brackets $x[n]$ for a discrete-time signal. When we talk about continuous time signals

$$f(t) \quad t \in \mathbb{R}, \tag{1.4}$$

we will always use curvy parenthesis "$(\cdot)$." It is important to note that a discrete time signal $x[n]$ is only defined for $n \in \mathbb{Z}$. In Figure 1, we don't imagine that the signal x takes on zero values off of the integers – the value $x[\frac{1}{2}]$ is simply not defined.[2]

Obtaining Discrete-Time Signals from Continuous-Time Signals. Many discrete-time signals $x[n]$ that are useful for applications can be viewed as sampled versions of a continuous time signal $x_a(t)$.[3] That is to say, we set

$$x[n] = x_a(nT) \qquad n \in \mathbb{Z}. \tag{1.5}$$

Here, T is the sampling period, with units of time (e.g., seconds). The sampling frequency is $f = 1/T$, with units of frequency (e.g., Hz). Figure 2 shows a continuous time signal, and its sampled version.

Often we are interested in processing the sampled signal $x[n]$ to learn something about the continuous signal $x_a(t)$. So, it is very important to understand the relationship between the two. In

[1] Imagine, for example, generating a measurement $x[n]$ by measuring the temperature at various points along a straight line. Here, n really means distance. However, when we talk in general about one-dimensional discrete-time signals, we prefer to refer to n as time.

[2] Actually, discrete-time signals are much simpler mathematical objects than continuous-time signals. In undergraduate signals and systems, you were probably given cryptic warnings about the fact that the Dirac delta $\delta(t)$ is not a function, but rather a functional/distribution. Most conceptual difficulties of this nature vanish when we work in discrete time.

[3] Here, the a in $x_a(t)$ stands for analog.

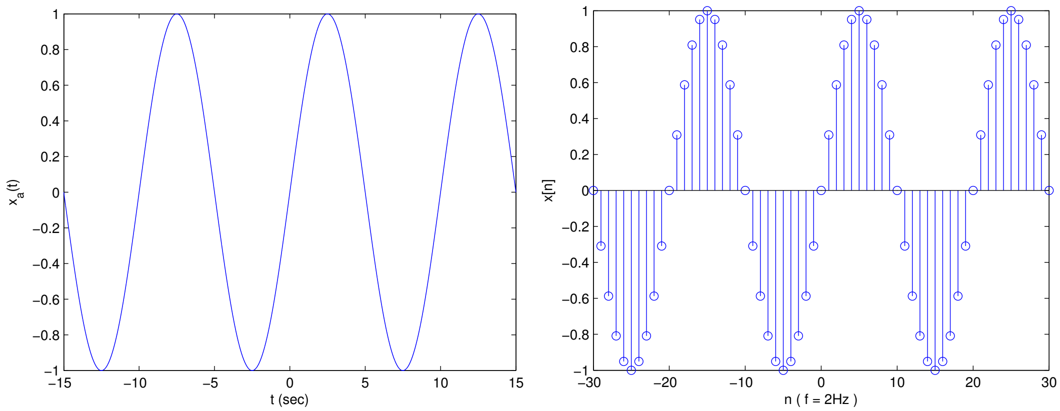

Figure 2: **Sampling a continuous time signal.** Left: continuous time signal $x_a(t)$. Right: sampled discrete-time signal $x[n]$.

some situations – for example, when $x_a(t)$ is *bandlimited*[4] – it is possible to exactly reconstruct $x_a(t)$ from the samples $x[n]$. Understanding when and how this is possible will be a major topic in the course.

What is $x[n]$? Thus far, we have been vague about the nature of the values $x[n]$ – we simply told you that they are "numbers". Depending on the application, $x[n]$ might be modeled as real numbers $x[n] \in \mathbb{R}$, or they might be quantized to live in some smaller set $x[n] \in \mathcal{Q}$. For example, many popular image representation formats use eight bits per pixel, and the signal is comprised of numbers $x[n] \in \mathcal{Q} = \{0, 1, 2, \ldots, 255\}$. Thus, the range of $x[n]$ is somewhat application-dependent. However, for developing analytical tools and insights, it is often very convenient to assume that our signal $x[n]$ is comprised of complex numbers $x[n] \in \mathbb{C}$.

2 Complex Arithmetic in Signal Processing

In signal processing, we often work over the complex numbers, rather than the reals. This may seem somewhat strange at first glance, since the mathematical models that we make for the world are often real-valued. For example, when we talk about sound, we talk about air pressure as a function of time. Air pressure can be plausibly modeled as a single real number. When we talk about images, we are talking about the number of photons that hit a given photosensitive surface, per unit time, as a function of space. Again, the number of photons hitting in a given time period is an integer; the rate at which photons impinge on the surface can be plausibly modeled as a real number.

So, why then, are complex numbers so prevalent in the analysis of signal processing systems? The answer is simple: complex numbers provide a convenient system for making calculations – in particular, for making calculations with *sinusoids*. Sinusoids are important for human perception

[4] A *bandlimited* signal is one whose Fourier transform vanishes outside of some interval $[-\Omega, \Omega]$ – put informally, "the signal only contains frequencies below Ω." We will review properties of the Fourier transform beginning in Lecture 3.

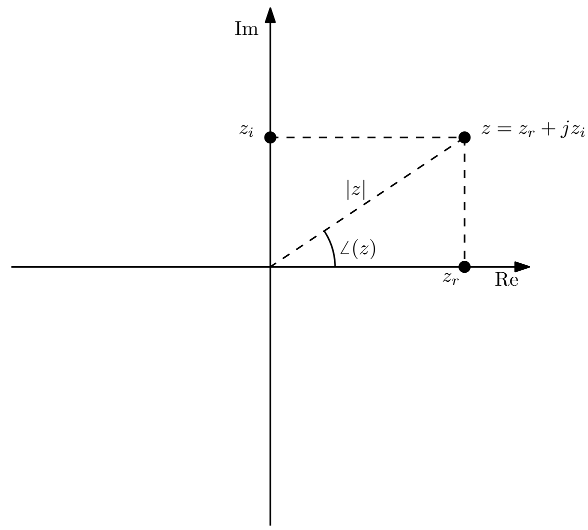

Figure 3: **Graphical representation of complex numbers**

– our ears are naturally attuned to distinguish oscillations at different frequency. They also play a very fundamental mathematical role, as the eigenfunctions of linear time-invariant systems.

Much of the utility of complex numbers derives from the magic of *Euler's formula*,

$$e^{j\theta} = \cos(\theta) + j\sin(\theta), \tag{2.1}$$

which we will re-derive below. This allows us to compute with sinusoids in a very convenient way, by using complex exponentials. Before reminding ourselves of this relationship, we briefly recall the properties of the complex numbers.

2.1 The complex numbers

The *complex numbers* $\mathbb{C}$ consist of pairs $z = (z_r, z_i)$, which are sometimes referred to as the real and imaginary parts. Following the convention in electrical engineering, we reserve the symbol j for the *imaginary unit* (or square root of -1). We often write the pair $z = (z_r, z_i)$ as

$$z = z_r + jz_i. \tag{2.2}$$

The complex number $z = z_r + jz_i$ can be visualized as a two dimensional vector, whose horizontal component is z_r and whose vertical component is z_i.

It is useful to have notation for the operators that extract the real and imaginary parts of the complex number z. For $z = z_r + jz_i$, we write

$$\mathrm{Re}[z] = z_r, \qquad \mathrm{Im}[z] = z_i. \tag{2.3}$$

The complex numbers form a *field* – that is to say, we can add them and multiply them in ways that conform to our intuition. In particular, if $w = w_r + jw_i \in \mathbb{C}$ and $z = z_r + jz_i \in \mathbb{C}$, then the sum $w + z$ is simply

$$w + z = (w_r + z_r) + j(w_i + z_i). \tag{2.4}$$

In terms of the vector picture, we simply add the vectors w and z tip-to-tail.

Complex multiplication works in a similarly intuitive way, as long as we remember that $j^2 = -1$. Namely,

$$\begin{aligned} wz &= (w_r + jw_i)(z_r + jz_i) \\ &= w_r z_r + jw_r z_i + jw_i z_r + j^2 w_i z_i \\ &= \underbrace{(w_r z_r - w_i z_i)}_{\text{real part}} + j \underbrace{(w_r z_i + w_i z_r)}_{\text{imaginary part}}. \end{aligned} \tag{2.5}$$

The length $|z| = \sqrt{z_r^2 + z_i^2}$ of the vector (z_r, z_i), is sometimes known as the magnitude or modulus of the complex number $\mathbb{C}$, and plays a very important role in studying sums of many complex numbers.

2.2 Infinite sums and convergence

If we have a sequence of k complex numbers $z_1, z_2, \ldots, z_k$, we can sum them:

$$z_1 + z_2 + \cdots + z_k. \tag{2.6}$$

It is sometimes more convenient to write this as

$$z_1 + z_2 + \cdots + z_k = \sum_{i=1}^{k} z_i. \tag{2.7}$$

Summability and infinite sequences. If the sequence $z_1, z_2, \ldots, z_i, \ldots$ is defined for every integer $i > 0$, we can define the partial summation

$$S_k = \sum_{i=1}^{k} z_i. \tag{2.8}$$

The infinite summation

$$\sum_{i=1}^{\infty} z_i = \lim_{k \to \infty} S_k, \tag{2.9}$$

is defined as the limit of the partial summations, *whenever the limit exists*.[5] If the limit exists, we call the sequence *summable*.

[5]It is easy to make examples for which the limit does not exist. Consider, e.g.,

$$z_i = \begin{cases} 1 & i \text{ even}, \\ -1 & i \text{ odd}. \end{cases}$$

Then

$$S_k = \begin{cases} 0 & k \text{ even}, \\ -1 & k \text{ odd}, \end{cases}$$

the limit does not exist, and $\sum_{i=1}^{\infty} z_i$ is not defined. These kinds of technicalities become important when we start to analyze signals in transform domains, such as the (discrete-time) Fourier domain, because the transforms of interest are defined via infinite summations. If the infinite summation is not defined, the transform is not defined for that input signal.

Absolute summability. It can be tricky to directly check whether the limit in (2.9) exists. We can give a simpler sufficient condition, based on the notion of *absolute summability*. For a sequence $z_1, z_2, \ldots,$ consider the summation

$$\sum_{i=1}^{\infty} |z_i|. \tag{2.10}$$

Because $|z_i| \geq 0$ for every i, either $\sum_{i=1}^{\infty} |z_i|$ is a nonnegative real number, or it is equal to $+\infty$. If

$$\sum_{i=1}^{\infty} |z_i| < \infty, \tag{2.11}$$

we say that the sequence $z_1, z_2, \ldots,$ is *absolute summable*. Any *absolute summable* sequence summable. So, whenever $\sum_{i=1}^{\infty} |z_i| < \infty$, the summation $\sum_{i=1}^{\infty} z_i$ exists.

Two-sided sequences. We will occasionally encounter two-sided sequences z_i which are defined for every $i \in \mathbb{Z}$. I.e., the sequence

$$\ldots, z_{-2}, z_{-1}, z_0, z_1, z_2, \ldots \tag{2.12}$$

extends infinitely both to the left and to the right. We would like to have a notion of summing this two-sided sequence – i.e., we would like to write

$$\sum_{i=-\infty}^{\infty} z_i. \tag{2.13}$$

The definition of such a two-sided infinite summation is a bit more technical.[6] However, again a simpler *sufficient condition* for the existence of the summation in (2.13) can be given using the notion of absolute summability. If

$$\sum_{i=-\infty}^{\infty} |z_i| < +\infty, \tag{2.17}$$

then $\sum_{i=-\infty}^{\infty} z_i$ is well-defined.

[6]**The following footnote is in red to indicate that it is here for your enjoyment and edification, but can be skipped at a first reading.** We define a two-sided partial summation

$$S_{k,k'} = \sum_{i=-k}^{k'} z_i. \tag{2.14}$$

The infinite summation

$$\sum_{i=-\infty}^{\infty} z_i \tag{2.15}$$

is defined whenever we can interchange taking the limit with respect to k and taking the limit with respect to k':

$$\lim_{k\to\infty} \lim_{k'\to\infty} S_{k,k'} = \lim_{k'\to\infty} \lim_{k\to\infty} S_{k,k'}. \tag{2.16}$$

Whenever these two limits exist and are equal, we define $\sum_{i=-\infty}^{\infty} z_i$ to be equal to the common value. A sufficient condition for this is that the sequence z_i is absolute summable, ala (2.17).

2.3 Euler's formula

The *exponential* of a complex number $x \in \mathbb{C}$ can be defined through the power series

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{2.18}$$

It is not too difficult to show that the infinite summation above is convergent for any $x \in \mathbb{C}$.

Theorem 2.1. *For any $\theta \in \mathbb{R}$, $e^{j\theta} = \cos(\theta) + j\sin(\theta)$.*

Proof. To understand why this is true, we recall the power series representations $\cos$ and $\sin$. Namely, for any $t \in \mathbb{R}$,

$$\begin{aligned} \cos(t) &= 1 - t^2/2! + t^4/4! - t^6/6! + \ldots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!}. \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \sin(t) &= t - t^3/3! + t^5/5! - t^7/7! + \ldots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}. \end{aligned} \tag{2.20}$$

Plugging in $t = j\theta$ to (2.18), and recalling that $j^2 = -1$, we see that

$$\begin{aligned} \exp(j\theta) &= 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \frac{(j\theta)^6}{6!} + \ldots \\ &= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \ldots \\ &= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) \quad + \quad \left(j\theta - j\frac{\theta^3}{3!} + j\frac{\theta^5}{5!} + \ldots\right) \\ &= \cos(\theta) + j\sin(\theta), \end{aligned} \tag{2.21}$$

as desired. □

2.4 Polar representation

Using Euler's formula, we can form a "polar" representation of the complex number $z = z_r + jz_i$. The *magnitude* is

$$|z| = \sqrt{z_r^2 + z_i^2}. \tag{2.22}$$

In the vector picture in Figure 3, the magnitude is the length of the vector (z_r, z_i). The *phase* is

$$\angle(z) = \tan^{-1}(z_i/z_r). \tag{2.23}$$

This is the angle between the vector (z_r, z_i) and the horizontal axis. In terms of these quantities, the polar representation is

$$z = |z|e^{j\angle(z)}. \tag{2.24}$$

The polar form is very convenient for multiplying complex exponentials. Given $z = |z|e^{j\angle(z)}$ and $w = |w|e^{j\angle(w)}$, the product zw is simply

$$zw = |z||w|e^{j(\angle(z)+\angle(w))}. \tag{2.25}$$

That is to say,

$$|zw| = |z||w|, \tag{2.26}$$

and

$$\angle(zw) = \angle(z) + \angle(w). \tag{2.27}$$

When multiplying complex numbers, magnitudes multiply and phases add.

2.5 Complex conjugation

The *conjugate* of a complex number $z = z_r + jz_i \in \mathbb{C}$ is given by

$$z^* = z_r - jz_i. \tag{2.28}$$

That is to say, we flip the sign of the imaginary part. Pictorally, we flip the vector (z_r, z_i) about the horizontal axis. It is also easy to check that after conjugation, the polar representation $z = |z|e^{j\theta(z)}$ becomes

$$z^* = |z|e^{-j\angle(z)}. \tag{2.29}$$

That is to say *conjugation does not change the magnitude, but multiplies the phase by* -1:

$$\angle(z^*) = -\angle(z). \tag{2.30}$$

From (2.29), we can see that

$$zz^* = |z|^2, \tag{2.31}$$

or equivalently, $|z| = \sqrt{zz^*}$. We can also notice that the product of a complex number z and its conjugate z^* is always real.

Similarly, the sum of z and z^* is always real:

$$z + z^* = z_r + jz_i \, + \, z_r - jz_i. \tag{2.32}$$

From this, we get

$$\mathrm{Re}[z] = \frac{z + z^*}{2}, \tag{2.33}$$

and

$$\mathrm{Im}[z] = \frac{z - z^*}{2j}. \tag{2.34}$$

If any of the above material on complex numbers is rusty, please take some time to review it. The textbook contains numerous worked problems with answers, which you can use to check your understanding.

3 Basic Signals

The unit impulse. The *unit impulse* $\delta[n]$ is defined as

$$\delta[n] = \begin{cases} 1 & n = 0, \\ 0 & \text{else.} \end{cases} \tag{3.1}$$

In contrast to the continuous-time Dirac delta, there are no serious technical complications in the definition of the unit impulse $\delta[n]$. To get an impulse at location k, we simply apply a shift:

$$\delta[n-k] = \begin{cases} 1 & n = k, \\ 0 & \text{else.} \end{cases} \tag{3.2}$$

We can express any input signal $x[n]$ as a superposition of shifted impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \tag{3.3}$$

This basic formula will be very useful next lecture when we study linear time invariant (LTI) systems and their impulse response – because any signal $x[n]$ can be constructed as a superposition of shifted unit impulses, we will be able to completely characterize an LTI system by studying how it responds to a unit impulse.

The unit step. The unit step function is

$$u[n] = \begin{cases} 0 & n < 0, \\ 1 & n \geq 0. \end{cases} \tag{3.4}$$

The unit step can be viewed as the "integral" of a unit impulse $\delta[\cdot]$:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]. \tag{3.5}$$

Please stop to think carefully about this formula! To verify that it is true, just check that the quantities on both sides agree for every choice of $n \in \mathbb{Z}$.

Please also take a moment to verify the following formula, which goes in the other direction and expresses $\delta[n]$ as a difference of unit steps:

$$\delta[n] = u[n] - u[n-1]. \tag{3.6}$$

Exponential sequences. An exponential sequence has the general form

$$x[n] = A\alpha^n. \tag{3.7}$$

When A and α are real, $x[n]$ either grows very rapidly in magnitude (if $|\alpha| > 1$) or decays very rapidly in magnitude (if $|\alpha| < 1$). If A and α are complex, the situation is more complicated. If their polar representations are

$$A = |A| \exp(j\phi) \tag{3.8}$$

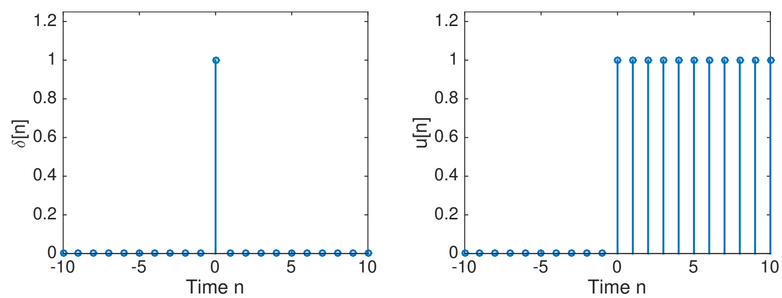

Figure 4: **Basic Sequences.** Left: the unit impulse (discrete delta), $\delta[n]$. Right: the unit step $u[n]$.

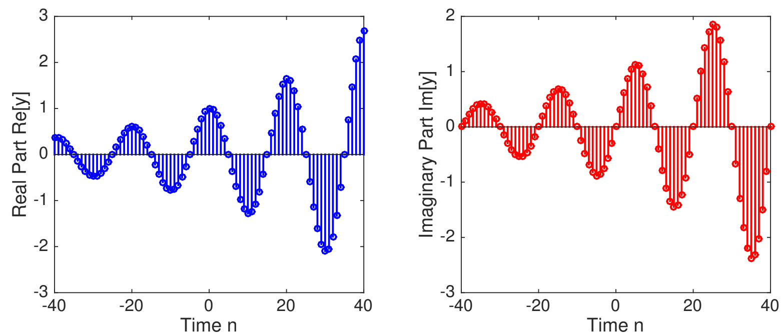

Figure 5: **An exponential sequence.** The sequence $y[n] = (1.025)^n \exp(j(\pi n/10))$. Left: real part $\mathrm{Re}[y]$. Right: imaginary part $\mathrm{Im}[y]$. The sequence consists of a complex exponential, windowed by the exponentially increasing sequence 1.025^n.

and

$$\alpha = |\alpha| \exp(j\omega) \tag{3.9}$$

then we can write

$$x[n] = A\alpha^n = |A||\alpha|^n \exp\Big(j(\omega n + \phi)\Big). \tag{3.10}$$

Using Euler's formula, we can further express this signal as

$$x[n] = |A||\alpha|^n \cos\Big(\omega n + \phi\Big) + j|A||\alpha|^n \sin\Big(\omega n + \phi\Big). \tag{3.11}$$

That is to say, the real and complex parts of $x[n]$ are sinusoids, with frequency ω, and phase shift ϕ. They are modulated by an exponentially increasing (or decreasing) sequence $|A||\alpha|^n$.

A discrete-time signal $x[n]$ is only defined for integers n. This means that the signals

$$x[n] = A \exp(j\omega n) \tag{3.12}$$

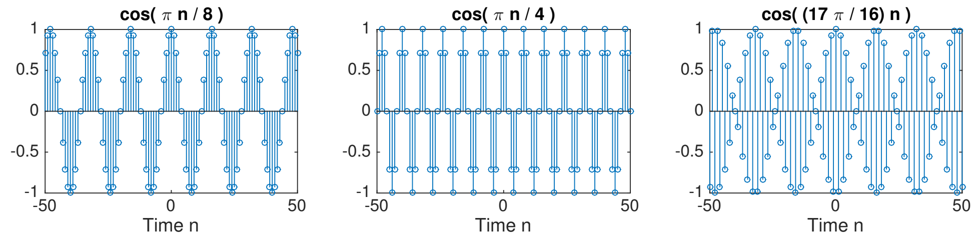

Figure 6: **Frequency in discrete time.** The cosine sequence $y[n] = \cos(\omega n)$ for three values of ω. Left: $\omega = \pi/8$. Middle: $\omega = \pi/4$. Right $\omega = 17\pi/16$. Notice that although $\pi/8 < \pi/4 < 17\pi/16$, the sequence at right appears to oscillate at a slower rate than the other two sequences. *Increasing ω only implies faster oscillation in certain intervals – e.g., when $0 \leq \omega < \pi$.*

and

$$\begin{aligned} y[n] &= A\exp(j(\omega + 2\pi)n) \\ &= A\exp(j\omega n)\exp(j2\pi n) \\ &= A\exp(j\omega n) \qquad \forall n \in \mathbb{Z} \qquad (3.13) \\ &= x[n] \qquad (3.14) \end{aligned}$$

are indistinguishable. So, *for discrete time complex exponentials and sinusoids, the frequency ω is only defined up addition by a multiple of 2π.* We usually restrict our attention to frequencies ω in an interval of length 2π: for example, $-\pi < \omega \leq \pi$, or $0 \leq \omega < 2\pi$.

Because of this ambiguity, we need to be careful about the interpretation of the frequency ω. In continuous time, higher frequency signals oscillate more quickly. In discrete time, if we consider a sinusoidal or complex exponential signal with frequency ω, say,

$$x[n] = \cos(\omega n), \qquad (3.15)$$

then as ω increases from 0 to π, the signal will oscillate more and more quickly. However, if we continue to increase ω beyond π, the signal $x[n]$ oscillates less quickly. When ω reaches 2π, the signal is actually constant! So, *in general, bigger ω does not imply faster oscillation – this only happens when ω belongs to certain intervals.*

Discrete-time periodic signals. A signal $x[n]$ is periodic with period N, if

$$x[n] = x[n+N], \qquad \forall n \in \mathbb{Z}. \qquad (3.16)$$

That is to say, every N samples, the signal repeats itself. Because n is restricted to be an integer, *discrete-time complex exponentials and sinusoids are not necessarily periodic*. A necessary and sufficient condition for a discrete time sinusoidal sequence to have period N is that the frequency ω is an integer multiple of $2\pi/N$:

Proposition 3.1. *The signals*

$$\begin{aligned} x_1[n] &= \exp\{j(\omega n + \phi)\} \qquad (3.17) \\ x_2[n] &= \cos(\omega n + \phi) \qquad (3.18) \\ x_3[n] &= \sin(\omega n + \phi) \qquad (3.19) \end{aligned}$$

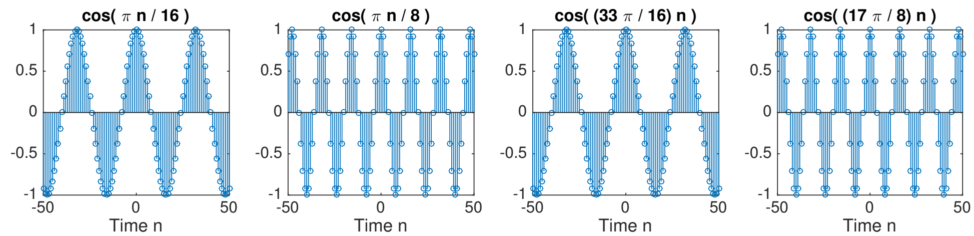

Figure 7: **Discrete-time frequency is only defined up to addition of 2π.** Sequences $y[n] = \cos(\omega n)$, for $\omega = \pi/16$, $\pi/8$, $\pi/16 + 2\pi = 33\pi/16$, $\pi/8 + 2\pi = 17\pi/8$. *Notice that adding 2π to the frequency ω does not change the sequence $y[n]$.*

are periodic with period N if and only if $\omega = 2\pi k/N$ for some integer $k \in \mathbb{Z}$.

Try to convince yourself of this fact. For completeness, I've sketched a proof in Appendix A.

Consider a complex exponential sequence $x[n] = \exp\{j(\omega n + \phi)\}$, and suppose that this sequence is periodic with period N. From the previous proposition, we know that $\omega = 2\pi k/N$ for some integer k. Moreover, for every integer L, the frequencies ω and $\omega + 2\pi L$ produce exactly the same sequence $x[n]$. So, we can assume that ω belongs to the set

$$\left\{0,\ \frac{2\pi}{N},\ \frac{2\pi}{N} \times 2,\ \ldots,\ \frac{2\pi}{N} \times (N-1)\right\}. \tag{3.20}$$

That is to say, *there are exactly N distinct frequencies ω of N-periodic complex exponentials*. More formally:

Proposition 3.2. *Suppose that $x[n] = \exp\{j(\omega n + \phi)\}$ is periodic, with period N. Then*

$$x[n] = \exp\{j(\omega_0 n + \phi)\} \tag{3.21}$$

for some ω_0 in the set

$$\Gamma = \left\{0,\ \frac{2\pi}{N},\ \frac{2\pi}{N} \times 2,\ \ldots,\ \frac{2\pi}{N} \times (N-1)\right\}. \tag{3.22}$$

This fact will be very important when we start working with finite length signals and the Discrete Fourier Transform (DFT).

A Appendix: Proof of Proposition 3.1

Proof. The "if" portion of the claim is straightforward. The "only if" requires a bit of calculation. Consider first the function $\sin(\omega n + \phi)$. Note that

$$\sin(\vartheta) = \sin(\varphi) \tag{A.1}$$

implies that either

$$\vartheta = \varphi + 2\pi M \tag{A.2}$$

or

$$\pi - \vartheta = \varphi + 2\pi M, \tag{A.3}$$

for some integer M. Periodicity implies that for all n,

$$\sin(\omega n + \phi) = \sin(\omega(n + N) + \phi). \tag{A.4}$$

If, for some n,

$$\omega n + \phi = \omega(n + N) + \phi + 2\pi M, \tag{A.5}$$

this implies that $\omega = -2\pi M/N$ is of the desired form. Suppose instead that for every n,

$$\pi - (\omega n + \phi) = \omega(n + N) + \phi + 2\pi M_n, \tag{A.6}$$

for some integer M_n. Evaluating at $n = 0$, we obtain that

$$2\phi = \pi - \omega N - 2\pi M_0. \tag{A.7}$$

Simplifying (A.6), we obtain

$$-\omega(2n + N) = -\pi + 2\phi + 2\pi M_n, \tag{A.8}$$

and plugging in gives

$$-\omega(2n + N) = -\omega N + 2\pi(M_n - M_0), \tag{A.9}$$

Evaluating at $n = 1$, we obtain

$$\omega = -\pi(M_n - M_0) \tag{A.10}$$

so ω is an integer multiple of π. Hence, there are essentially two possible frequencies ω: $\omega = 0$, and $\omega = \pi$. $\omega = 0$ has the requisite form. Notice that if N is odd, $\omega = \pi$ does not yield an N-periodic signal, and hence is impossible, while if N is even, $\omega = \pi$ has the requisite form. This completes the proof for $\sin$. Using that $\cos\theta = \sin(\pi/2 - \theta)$, this implies the claim for $\cos$. The results for $\cos$ and $\sin$ imply the claim for general complex exponentials, via Euler's formula. □

B Symmetries of Complex Signals

In this appendix, we briefly define several classes of complex signals, including the symmetric, conjugate symmetric, and conjugate antisymmetric signals. Symmetric signals rarely occur in nature – this is why it is banished to an appendix of these notes! However, the notion is useful in analyzing systems. For example, a conjugate symmetric signal has a real Fourier transform. When we get to FIR filter design, we will design filters with certain symmetries (or anti symmetries) that achieve some desired effect in the frequency domain.

Symmetric, and conjugate symmetric signals. We say that a signal $x[n]$ is *symmetric* (or *even*) if

$$x[n] = x[-n]. \tag{B.1}$$

That is to say, the signal is unchanged if we flip it about zero. We say that a signal is *antisymmetric*, or *odd*, if

$$x[n] = -x[-n]. \tag{B.2}$$

That is to say, flipping the signal about zero produces its negative.

When we deal with complex signals $x \in \mathbb{C}^{\mathbb{Z}}$, it is often useful to consider the effect of both flipping the signal about $n = 0$, and taking the complex conjugate. We say that a signal $x[n]$ is *conjugate symmetric* if

$$x[n] = x^*[-n], \tag{B.3}$$

Check for yourself that $x[n]$ is conjugate symmetric if and only if $\mathrm{Re}[x]$ is an even sequence and $\mathrm{Im}[x]$ is an odd sequence.

We say that $x[n]$ is *conjugate antisymmetric* if

$$x[n] = -x^*[-n]. \tag{B.4}$$

A sequence $x[n]$ is conjugate antisymmetric if and only if $\mathrm{Re}[x]$ is an odd sequence and $\mathrm{Im}[x]$ is an even sequence.

It is not particularly common to encounter (conjugate) symmetric signals in nature. However, we will see that the definition is still very useful for talking about properties of the Fourier and Z transforms.[7] We can express any given input signal as a sum of a conjugate symmetric and conjugate antisymmetric signal:

Proposition B.1. *Every signal $x[n]$ can be written as a sum*

$$x[n] = x_{ca}[n] + x_{cs}[n] \tag{B.5}$$

of a conjugate symmetric signal $x_{cs}[n]$ and a conjugate anti-symmetric signal $x_{ca}[n]$.

Proof. Write

$$x_{cs}[n] = \frac{x[n] + x^*[-n]}{2} \tag{B.6}$$

and

$$x_{ca}[n] = \frac{x[n] - x^*[-n]}{2}. \tag{B.7}$$

Notice that $x^*_{cs}[-n] = x_{cs}[n]$, so x_{cs} is indeed conjugate-symmetric, while $x^*_{ca}[-n] = -x_{ca}[n]$, so x_{ca} is indeed conjugate anti-symmetric. Finally, observe that

$$x_{cs}[n] + x_{ca}[n] = \frac{1}{2}\Big(x[n] + x^*[-n] + x[n] - x^*[-n]\Big) = x[n], \tag{B.8}$$

as desired. □

[7] For example, you may recall that a (continuous-time) signal $f(t)$ is real valued if and only if its (continuous-time) Fourier transform is conjugate symmetric.